

# Birefringence in nonlinear anisotropic dielectric media

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Light propagation is investigated in the context of local anisotropic nonlinear dielectric media at rest with the dielectric coefficients  $\varepsilon^{\mu}{}_{\nu} = \varepsilon^{\mu}{}_{\nu}(\vec{E}, \vec{B})$  and constant  $\mu$ , in the limit of geometrical optics. Birefringence was examined and the general conditions for its occurrence were presented. A toy model is exhibited, in which uniaxial birefringent media with nonlinear dielectric properties could be driven by external fields in such way that birefringence may be artificially controlled. The effective geometry interpretation is also addressed.

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## I. INTRODUCTION

In the regime of intense electromagnetic fields or inside some material media the equations governing electrodynamic phenomena are nonlinear. In the first case, the theory is built from a nonlinear Lagrangian [1] which is a function of the two Lorentz invariants of the electromagnetic field. In the second case, Maxwell equations must be supplemented with constitutive relations between external applied fields and the induced excitations. In general, such relations are nonlinear (although linear constitutive relations are also of interest [2]), and they depend on the physical properties of each considered medium under the action of external fields. In both cases the field equations will be presented in a nonlinear form and, as a consequence, several non usual effects (in the context of Maxwell theory) are predicted.

Of particular interest is the phenomenon of birefringence: the light velocity dependence on the polarization mode of the propagating wave. In the case of nonlinear electrodynamics, birefringence occurs when the electric and/or magnetic fields exceed its critical value predicted by quantum electrodynamics [3]. The analysis of light propagation in such regime shows that there is a probability of photon splitting under a strong external electromagnetic field [4]. More recent investigations on light propagation in the context of nonlinear Lagrangian for electrodynamics can be found in Refs. [5, 6, 7, 8] and references therein. Particularly, in Ref. [8] the Fresnel analysis of wave propagation in nonlinear electrodynamics was performed. For a class of local nonlinear Lagrangian nondispersive models, birefringence phenomena were studied and the wave propagation in a moving isotropic nonlinear medium was also examined.

Natural uniaxial birefringence is a well known effect [9] and it takes place for some materials (mostly crystals). Artificial birefringence is also possible to occur

[10, 11, 12, 13], as an induced effect in material media: when an external field is applied in a medium with nonlinear dielectrics properties, an artificial optical axis may appear. Birefringence is also used as a technique for investigating other properties of some systems: see for instance Ref. [14], where the birefringence effect is used as a tool for astrophysical studies.

In this work, birefringence is investigated in the context of homogeneous dielectric media at rest with the dielectric coefficients  $\varepsilon^{\mu}{}_{\nu} = \varepsilon^{\mu}{}_{\nu}(\vec{E}, \vec{B})$  and constant  $\mu$  in the limit of geometrical optics. The analysis is restricted to local electrodynamical models, and dispersive effects were neglected by considering only monochromatic waves, thus avoiding ambiguities with the velocity of the wave. A mechanism by which birefringence can be controlled by means of an external electric field is proposed. In particular, it is shown that naturally uniaxial media presenting nonlinear dielectric properties can be operated by external fields in such way that birefringence could be artificially turned off.

In the following section, the generalized eigen-vector equation associated with the light propagation in general anisotropic media is presented. In Sec. III the solution for such equation is presented for the particular case where the impermeability  $\mu^{\mu}{}_{\nu}$  is assumed to be a constant, and birefringence conditions are examined. Section IV particularizes to the case  $\varepsilon^{\alpha}{}_{\beta}(E)$ , and the principal refractive indices are explicitly presented. In Sec. V a particular model is exhibited in which a naturally uniaxial media presenting nonlinear dielectric properties, under the action of an external electric field, behaves in such way that birefringence could to be artificially controlled.

A covariant formalism is used throughout this work. Spacetime is assumed to be Minkowskian, and a Cartesian coordinate system is used, such that the metric is  $\eta_{\mu\nu} = \text{diag}(+1, -1, -1, -1)$ . Units are chosen such that  $c = 1$ , except mentioned otherwise. A geodetic observer  $V^{\mu} = \delta_0^{\mu}$  is supposed to describe all quantities. Particularly the electric field is represented by  $E^{\mu} = -F^{\mu\nu}V_{\nu} = (0, \vec{E})$  whose modulus is  $E = (-E^{\alpha}E_{\alpha})^{1/2}$ , and similarly for the magnetic field  $B^{\mu} = (0, \vec{B})$ .

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## II. WAVE EQUATION

The electrodynamics in a medium at rest is completely determined by the Maxwell equations

$$V^\mu D^\alpha_{,\mu} + \eta^{\alpha\beta\gamma\delta} V_\gamma H_{\delta,\beta} = 0 \quad (1)$$

$$V^\mu B^\alpha_{,\mu} - \eta^{\alpha\beta\gamma\delta} V_\gamma E_{\delta,\beta} = 0, \quad (2)$$

together with the constitutive relations

$$D^\alpha = \varepsilon^\alpha_\beta E^\beta \quad (3)$$

$$H^\alpha = \mu^\alpha_\beta B^\beta, \quad (4)$$

where the coefficients  $\varepsilon^\alpha_\beta = \varepsilon^\alpha_\beta(\vec{E}, \vec{B})$  and  $\mu^\alpha_\beta = \mu^\alpha_\beta(\vec{E}, \vec{B})$  represent the dielectric tensors (called permittivity and impermeability tensors, respectively) which encompass all information about the dielectric properties of the medium. They are usually assumed as being frequency-dependent, although they may more generally depend on the external fields themselves. The coordinate derivatives of Eqs. (3) and (4) are

$$D^\alpha_{,\mu} = \varepsilon^\alpha_\beta E^\beta_{,\mu} + \frac{\partial \varepsilon^\alpha_\beta}{\partial E^\tau} E^\beta E^\tau_{,\mu} + \frac{\partial \varepsilon^\alpha_\beta}{\partial B^\tau} E^\beta B^\tau_{,\mu} \quad (5)$$

$$H^\alpha_{,\mu} = \mu^\alpha_\beta B^\beta_{,\mu} + \frac{\partial \mu^\alpha_\beta}{\partial E^\tau} B^\beta E^\tau_{,\mu} + \frac{\partial \mu^\alpha_\beta}{\partial B^\tau} B^\beta B^\tau_{,\mu}. \quad (6)$$

In order to determine the propagation of the electromagnetic waves, we will consider the eikonal approximation of electrodynamics, making use of the method of field discontinuities [1, 15]. With the notation introduced in Ref. [11], we set  $[E^\mu]_\Sigma = e^\mu K_\nu$  and  $[B^\mu]_\Sigma = b^\mu K_\nu$ , where  $e^\mu$  and  $b^\mu$  represent the polarization of the wave fields on the surface of discontinuity  $\Sigma$  and  $K_\lambda = \partial\Sigma/\partial x^\lambda$  is the wave vector normal to  $\Sigma$ . By applying these boundary conditions to the Eqs. (1) and (2), and taking into account Eqs. (5) and (6), we obtain  $b^\mu = \eta^{\mu\nu\lambda\sigma} K_\nu V_\lambda e_\sigma / \omega$ , and

$$\left( \omega C^\alpha_\tau + \frac{\partial \mu_{\delta\lambda}}{\partial E^\tau} \eta^{\alpha\beta\gamma\delta} B^\lambda V_\gamma K_\beta + \frac{1}{\omega} H_\delta^\lambda \eta^{\alpha\beta\gamma\delta} \eta_{\lambda\lambda\nu\tau} V_\gamma K^\lambda V^\nu K_\beta \right) e^\tau = 0, \quad (7)$$

where we introduced the new quantities

$$C^\alpha_\tau \doteq \varepsilon^\alpha_\tau + \frac{\partial \varepsilon^\alpha_\beta}{\partial E^\tau} E^\beta + \frac{1}{\omega} \frac{\partial \varepsilon^\alpha_\beta}{\partial B^\rho} \eta^{\rho\lambda\gamma} E^\beta K_\lambda V_\gamma, \quad (8)$$

$$H^\alpha_\tau \doteq \mu^\alpha_\tau + \frac{\partial \mu^\alpha_\beta}{\partial B^\tau} B^\beta, \quad (9)$$

with  $\omega \doteq K^\alpha V_\alpha$  the frequency of the electromagnetic wave.

In order to present Eq. (7) in a 3-dimensional representation we introduce the projector on the 3-space  $h^\alpha_\mu = \delta^\alpha_\mu - V^\alpha V_\mu$ , and define

$$q^\alpha = h^\alpha_\mu K^\mu = K^\alpha - \omega V^\alpha. \quad (10)$$

Straightforward simplifications lead Eq. (7) to the form

$$Z^\alpha_\tau e^\tau = 0, \quad (11)$$

where

$$Z^\alpha_\tau = C^\alpha_\tau + \frac{q}{\omega} \frac{\partial \mu_{\delta\lambda}}{\partial E^\tau} \eta^{\alpha\beta\gamma\delta} B^\lambda V_\gamma \hat{q}_\beta + \frac{q^2}{\omega^2} I_\beta^{[\alpha} I_\tau^{\lambda]} H_\lambda^{\beta]} \quad (12)$$

in which  $a^{[\mu\nu]} = a^{\mu\nu} - a^{\nu\mu}$  for an arbitrary tensor  $a^{\mu\nu}$ . Further, we have defined the projector

$$I_\tau^\alpha = h_\tau^\alpha + \hat{q}^\alpha \hat{q}_\tau, \quad (13)$$

on the 2-space orthogonal to the propagation direction  $\hat{q}^\mu \doteq q^\mu / q$ , where  $q^2 = -h_{\mu\nu} q^\mu q^\nu$ .

The general solution for the wave propagation can be derived from the eigen-vector problem stated by Eq. (11), and is given by the generalized Fresnel equation

$$\det |Z^\alpha_\beta| = 0. \quad (14)$$

For several physical configurations [6] the dispersion relations coming from Eq. (14) can be written in the suggestive form  $g_\pm^{\mu\nu} K_\mu^\pm K_\nu^\pm = 0$ , as shown in Sec. III.

## III. BIREFRINGENCE CONDITIONS

Let us assume the more specific case where the medium behaves in such way that  $\mu^\alpha_\beta = \mu^{-1} h^\alpha_\beta$  with  $\mu$  a constant. In this case, Eq. (12) reduces to

$$Z^\alpha_\tau = C^\alpha_\tau - \frac{1}{\mu v_\varphi^2} I^\alpha_\tau, \quad (15)$$

where  $v_\varphi = \omega / q$  is the phase velocity of the wave. Equation (14) can be written as  $(Z^\mu_\mu)^3 - 3Z^\mu_\mu Z^\alpha_\beta Z^\beta_\alpha + 2Z^\alpha_\beta Z^\beta_\gamma Z^\gamma_\alpha = 0$ , which yields a fourth order equation of the form

$$\alpha v_\varphi^4 - \beta v_\varphi^2 - \gamma = 0, \quad (16)$$

where we have defined

$$\alpha \doteq \frac{1}{6} [(C^\mu_\mu)^3 - 3C^\mu_\mu C^\alpha_\beta C^\beta_\alpha + 2C^\alpha_\beta C^\beta_\gamma C^\gamma_\alpha], \quad (17)$$

$$\beta \doteq \mu^{-1} (C^\lambda_\alpha C^{\alpha\nu} - C^\alpha_\alpha C^{\lambda\nu}) \hat{q}_\lambda \hat{q}_\nu, \quad (18)$$

$$\gamma \doteq \mu^{-2} C^{\lambda\nu} \hat{q}_\lambda \hat{q}_\nu. \quad (19)$$

The solutions of Eq. (16) are

$$v_\varphi^\pm = \sqrt{\frac{\beta}{2\alpha} \left( 1 \pm \sqrt{1 + \frac{4\alpha\gamma}{\beta^2}} \right)}. \quad (20)$$

Equation (20) expresses the fact that, in general, the velocity of the electromagnetic waves inside a material medium may get two possible values  $v_\varphi^+$ ,  $v_\varphi^-$  which are associated to the two possible polarizations modes [12].

Thus, birefringence occurs if the following inequalities hold for a given direction  $\hat{q}^\lambda$ :

$$-\frac{1}{4} < \frac{\alpha\gamma}{\beta^2} < 0, \quad (21)$$

$$\alpha\beta > 0. \quad (22)$$

(For the particular case  $\varepsilon^\alpha_\beta = \epsilon h^\alpha_\beta$ , with  $\epsilon$  a constant, one obtains  $\alpha = \epsilon^3$ ,  $\beta = 2\epsilon^2/\mu$  and  $\gamma = -\epsilon/\mu^2$ , and the two phase velocities reduce both to the same well-known value  $v_\varphi^2 = 1/\epsilon\mu$ .)

### A. Effective geometry

The dispersion relation Eq. (16) can be conveniently expressed in the form  $g_\pm^{\mu\nu} K_\mu^\pm K_\nu^\pm = 0$  (corresponding to the factorization of the general fourth-order Fresnel tensor [16]), where

$$g_\pm^{\alpha\beta} = \mu\alpha V^\alpha V^\beta + \frac{1}{2} \left[ C^\nu_\nu - \frac{1}{\mu(v_\varphi^\pm)^2} \right] C^{(\alpha\beta)} - \frac{1}{2} C^{(\alpha}_\nu C^{\nu\beta)} \quad (23)$$

The symmetric tensors  $g_\pm^{\mu\nu}$  represent the effective geometries (also known as optical metrics) associated with the wave propagation, and the symbol  $\pm$  indicates that, in general, there will be two possible distinct metrics, one for each polarization mode. The integral curves of the vectors  $K_\mu^\pm$  are geodesics in the corresponding effective geometry  $g_\pm^{\mu\nu}$  [12]. For the particular case of the linear theory in vacuum, both  $g_+^{\mu\nu}$  and  $g_-^{\mu\nu}$  reduce to  $\eta^{\mu\nu}$  as expected. For the case of naturally isotropic nonlinear media at rest the results presented in Refs. [8, 11, 12] are recovered.

The effective geometry interpretation for electromagnetic waves in material media [11, 12, 17] (or even in the context of nonlinear electrodynamics in vacuum [5, 6, 18]) has been proposed as a tool for testing kinematic aspects of general relativity in laboratory [19]. In this context, Eq. (23) provides a natural support for analogue anisotropic models.

## IV. NON-MAGNETIC ANISOTROPIC NONLINEAR MEDIA

Let us consider a naturally uniaxial medium that reacts nonlinearly when subjected to an external electric field as  $\varepsilon^\alpha_\beta = \text{diag}[0, \varepsilon_\parallel(E), \varepsilon_\perp(E), \varepsilon_\perp(E)]$ . In this case,  $\varepsilon^\alpha_\beta = \varepsilon^\alpha_\beta(E)$  and Eq. (8) takes the simpler form

$$C^\alpha_\beta = \varepsilon^\alpha_\beta - \frac{\partial \varepsilon^\alpha_\tau}{\partial E} \frac{E^\tau E_\beta}{E}. \quad (24)$$

By setting  $\vec{E}$  in the  $x$ -direction (optical axis), we obtain  $C^\alpha_\beta = \text{diag}(0, \varepsilon_\parallel + E\varepsilon'_\parallel, \varepsilon_\perp, \varepsilon_\perp)$ , where  $\varepsilon'_\parallel = d\varepsilon_\parallel/dE$ . Now, from the above results, we obtain

$$\alpha\gamma = -\frac{\varepsilon_\perp^2 C^1_1}{\mu^2} [\varepsilon_\perp(1 - \hat{q}_1^2) + C^1_1 \hat{q}_1^2], \quad (25)$$

$$4\alpha\gamma + \beta^2 = \left[ \frac{\varepsilon_\perp}{\mu} (C^1_1 - \varepsilon_\perp)(1 - \hat{q}_1^2) \right]^2, \quad (26)$$

$$\alpha\beta = \frac{\varepsilon_\perp^3}{\mu} C^1_1 [C^1_1(1 + \hat{q}_1^2) + \varepsilon_\perp(1 - \hat{q}_1^2)] > 0, \quad (27)$$

which show that conditions (21) and (22) are fulfilled in this case, provided that  $C^1_1 \neq \varepsilon_\perp$  and  $\hat{q}_1^2 < 1$ , since both  $C^1_1$  and  $\varepsilon_\perp$  are positive quantities.

Equation (26) shows that birefringence can be annulled if  $C^1_1 = \varepsilon_\perp$ . In other words, birefringence phenomena could be turned off by imposing that

$$\varepsilon_\parallel + E\varepsilon'_\parallel = \varepsilon_\perp, \quad (28)$$

which is a *no-birefringence condition*. Note that if a model for dielectric permittivity is set, the no-birefringence condition will mean a condition on the value of the external electric field.

### A. Ordinary and extraordinary rays

By considering the previous model, Eq. (20) yields

$$(v_\varphi^\pm)^2 = \frac{1}{2\mu\varepsilon_\perp C^1_1} [C^1_1(1 + \hat{q}_1^2) + \varepsilon_\perp(1 - \hat{q}_1^2) \pm (C^1_1 - \varepsilon_\perp)(1 - \hat{q}_1^2)]. \quad (29)$$

Thus, there will be two phase velocities, one of which is isotropic (ordinary ray)

$$(v_\varphi^+)^2 = \frac{1}{\mu\varepsilon_\perp}, \quad (30)$$

and the other one depends on the direction of propagation (extraordinary ray)

$$(v_\varphi^-)^2 = \frac{1}{\mu\varepsilon_\perp C^1_1} [\varepsilon_\perp(1 - \hat{q}_1^2) + C^1_1 \hat{q}_1^2]. \quad (31)$$

The two velocities coincide when either the propagation occurs along the direction of the electric field, or when the no-birefringence condition Eq. (28) holds.

The two extreme values of  $v_\varphi^-$  occur in the direction of the electric field ( $\theta = 0$ ), and perpendicularly to it ( $\theta = \pi/2$ ). For these two directions, we obtain

$$v_\parallel^2 \doteq (v_\varphi^-)^2|_{\theta=0} = \frac{1}{\mu\varepsilon_\perp} = (v_\varphi^+)^2, \quad (32)$$

$$v_\perp^2 \doteq (v_\varphi^-)^2|_{\theta=\pi/2} = \frac{1}{\mu C^1_1}. \quad (33)$$

Associated to these directions we define the principal effective refractive indexes  $n_o$  and  $n_e$  by

$$n_o^2 = \frac{1}{v_\parallel^2} = \mu\varepsilon_\perp, \quad (34)$$

$$n_e^2 = \frac{1}{v_\perp^2} = \mu(\varepsilon_\parallel + E\varepsilon'_\parallel). \quad (35)$$

The effective refractive index of the extraordinary ray along an arbitrary direction  $\hat{q}^\lambda$  is determined by  $n_q = 1/v_\varphi^-$ , with  $v_\varphi^-$  given by Eq. (31).

## V. CONTROLLED BIREFRINGENCE

Let us examine some special cases where birefringence could be controlled by adjusting the magnitude of an external electric field.

### A. Non-birefringent nonlinear media

We consider the model where  $\epsilon_{\perp} = \epsilon_{\perp} - 3pE^2$ , with  $\epsilon_{\perp}$  and  $p$  constants. Thus, from Eq. (28) we obtain  $\epsilon_{\parallel} + E\epsilon'_{\parallel} = \epsilon_{\perp} - 3pE^2$ . Regularity on the solution of this equation at  $E = 0$  yields

$$\epsilon_{\parallel} = \epsilon_{\perp} - pE^2. \quad (36)$$

It means that for every value of  $E$ , if the medium reacts like Eq. (36), there won't be birefringence phenomena.

### B. Anisotropic media with artificially controlled birefringence

Now, let us examine the model for which the medium reacts as

$$\epsilon_{\perp} = \epsilon_{\perp} - 3pE^2, \quad (37)$$

$$\epsilon_{\parallel} = \epsilon_{\parallel} - sE^2, \quad (38)$$

with  $\epsilon_{\perp}$ ,  $p$ ,  $\epsilon_{\parallel}$  and  $s$  constants. For the case  $p = s$ , all values of  $E$  will correspond to either a no-birefringent configuration (if  $\epsilon_{\perp} = \epsilon_{\parallel}$ ), as described by Eq. (36), or else a birefringent configuration (if  $\epsilon_{\perp} \neq \epsilon_{\parallel}$ ). On the other hand, for  $p \neq s$ , Eq. (28) yields  $\epsilon_{\parallel} - 3sE^2 = \epsilon_{\perp} - 3pE^2$ , and solving it for the electric field one obtains

$$E_c^2 = \frac{\epsilon_{\parallel} - \epsilon_{\perp}}{3(s - p)}. \quad (39)$$

Thus, from Eq. (39) we conclude that there will be a particular value  $E_c$  of the electric field for which the no-birefringence condition will be satisfied. Any other value of  $E \neq E_c$  will result in birefringence. Of course these results are only of interest in the case  $E_c$  lies within the limit of applicability of Maxwell theory. Further, the quadratic terms in  $E$  in Eqs. (37) and (38) should remain small when compared with the constant ones. Equation (39) is meaningful for positive media ( $\epsilon_{\perp} < \epsilon_{\parallel}$ ) if  $p < 0$  and  $s > 0$ ; and similarly for negative media ( $\epsilon_{\perp} > \epsilon_{\parallel}$ ) if  $p > 0$  and  $s < 0$ .

For the case where  $\epsilon_{\parallel} \neq \epsilon_{\perp}$  birefringence will occur provided that  $E \neq E_c$ . For the particular case where  $E = 0$  we have natural birefringence, as it occurs in some crystals. See Fig. 1 for a numerical analysis of a toy model where natural birefringence is artificially controlled by the influence of an external electric field.

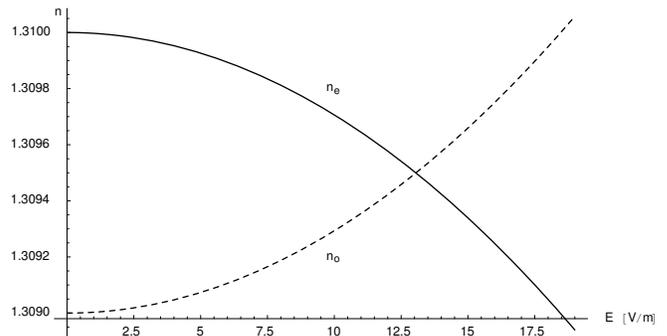


FIG. 1: Controlled birefringence: the particular value  $E_c$  of the electric field for which birefringence is turned off occurs in the intersection of the curves representing the effective refractive indices for the ordinary and extraordinary rays. The naturally uniaxial positive crystal becomes negative when the electric field exceeds  $E_c$ . Note that, for  $E = 0$ , natural birefringence remains. We used the following illustrative values (in MKS units) for the parameters appearing in Eqs. (37) and (38):  $\epsilon_{\parallel} = 1.5195 \times 10^{-11} F/m$ ,  $\epsilon_{\perp} = 1.5172 \times 10^{-11} F/m$ ,  $p = -2.2648 \times 10^{-17} Fm/V^2$  and  $s = 2.2700 \times 10^{-17} Fm/V^2$ .

By considering a naturally isotropic medium where  $\epsilon_{\parallel} = \epsilon_{\perp}$ , the model set by Eqs. (37) and (38) describes artificial birefringence [11] as it appears in the Kerr electro-optic effect. For this case we obtain  $n_o - n_e \approx \lambda K E^2$ , where we have defined  $\lambda K \doteq 3\sqrt{\mu\epsilon_{\perp}}(s - p)/2\epsilon_{\perp}$ , with  $\lambda$  the wave length of the electromagnetic waves and  $K$  the Kerr's constant (its value depends on the dielectric properties of the medium). The effective geometry for this particular case has already been presented in the literature [8, 11, 12].

## VI. CONCLUSION

A covariant tensorial formalism is here provided in order to discuss the propagation of monochromatic electromagnetic waves inside naturally anisotropic material media with nonlinear dielectric properties, in the limit of geometrical optics. The eigen-vector problem for general media was presented. For the case of constant  $\mu$ , the conditions for birefringence phenomena to appear were examined. For the case of non-magnetic media, a particular model was shown for which the magnitude of the birefringence could be controlled, and even turned off, with the use of an external electric field. The model generalizes the standard description of the Kerr electro-optical effect to the case of a naturally anisotropic material media. All the obtained results can be straightforwardly rewritten to the case  $\epsilon^{\alpha}_{\beta} = \epsilon h^{\alpha}_{\beta}$  with  $\epsilon = const$  and  $\mu^{\alpha}_{\beta} = \mu^{\alpha}_{\beta}(\vec{B})$ , thus generalizing the Cotton-Mouton effect (magnetic analogue to the Kerr effect) to the case of

anisotropic media.

The results can also be used in the context of analogue models for general relativity. It is here presented the optical metric for nondispersive anisotropic media, which extends previous investigations. Isotropic moving media were already discussed [8], and their results agree with ours in the case of isotropic media at rest. Anisotropic moving media are still to be considered.

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